Recursive Formulas for the Evaluation of the Convolution Integral

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ABSTRACT. Recursive formulas for the numerical evaluation of the real convolution integral are derived for the case in which the impulse response is given analytically. These formulas require considerably less computation time and memory space than the general time series formulas and can be effectively applied for digital simulation of continuous physical systems.

KEY WORDS AND PHRASES: convolution integral, numerical convolution, numerical integration, recursion formula, time series, transients, transfer function, digital simulation, continuous dynamics, ordinary differential equations

CR CATEGORIES: 5.13, 5.16, 5.17, 5.18

1. *Introduction*

The real convolution integral is a valuable mathematical tool for simulating control systems and electrical networks on a digital computer [1, 4]. For a linear system the convolution integral describes, via the impulse response $w(t)$, the relationship between the input signal $x(t)$ and the output signal $y(t)$. The convolution integral is written as

$$y(t) = \int_0^t x(\tau)w(t-\tau)\,d\tau \tag{1}$$

with

$$w(t) = 0 \qquad \text{for } t < 0.$$

This integral can be approximated by the following time series

$$\mathrm{y}(0) = 0$$

and

$$y(n\cdot\Delta t) = y_n = \Delta t \sum_{m=0}^{n} b_m x_m w_{n-m} \qquad \text{for } n > 1 \tag{2}$$

where x_m is the mth sample of input signal $x(t)$, w_{n-m} is the $(n-m)$-th sample of the impulse response $w(t)$, and Δt is the sampling interval. The coefficients b_m depend on the interpolation selected.

There are two major disadvantages to evaluating the integral by eq. (2): (a) the number of arithmetic operations increases with time; (b) all past values of $x(t)$ and $w(t)$ must be known.

If the impulse response $w(t)$ is available in the form

$$w(t) = \sum_{v=1}^{A} w_v(t) \tag{3a}$$

and

$$w_v(t) = B_v \exp(-C_v t) t^{D_v} \cos(E_v t + F_v), \tag{3b}$$

where D_v is an integer, then simple recursion formulas for the calculation of the output signal $y(n \cdot \Delta t)$ can be derived. Because the formulas are short, and also because the number of arithmetic operations involved is time-independent, their computer requirements for storage space and running time are less than those of eqs. (2).

In most instances the impulse response $w(t)$ is either obtained easily from transfer function tables [2, 3] in form (3) or given by measured data. In the latter case, form (3) can be obtained by curve fitting. Therefore we assume $w(t)$ to have form (3).

2. *Derivation of Recursion Formulas*

We can derive recursion formulas for the general impulse response $w(t)$ of (3) more easily if we first consider two special cases of $w(t)$. The first special case considered is the situation most frequently encountered in actual practice.

Case 1. Suppose D_v of eq. (3b) is zero, i.e. no multiple poles.
Case 2. Suppose C_v, E_v, and F_v of eq. (3b) are zero, i.e. multiple integration.

To further simplify the derivation of the recursion formulas, we separate the previous samples from the current samples. Let

$$y_n = \sum_{v=1}^{A} y_{vn} \tag{4a}$$

with

$$y_{vn} = P_{vn} + Q_v x_n \tag{4b}$$

and

$$Q_v = \Delta t\, b_n w_{v0} \tag{4c}$$

$$w_{v0} = \begin{cases} B_v \cos F_v & \text{for } D_v = 0, \\ 0 & \text{for } D_v \neq 0, \end{cases} \tag{4d}$$

where

$$P_{vn} = \Delta t \sum_{m=0}^{n-1} b_m x_m w_{v,n-m} \tag{4e}$$

contains all previous samples. When linear interpolation is applied the coefficient b_n has the value $\frac{1}{2}$, but when quadratic interpolation is applied b_n has the value $\frac{4}{3}$ for odd n amd $\frac{1}{3}$ for even n.

2.1 Recursion Formulas for Case 1

If we assume that the time intervals Δt are of the same length, then

$$w_v(n\Delta t) = w_{vn} = B_v \exp(-C_v n\Delta t) \cos(E_v n\Delta t + F_v). \tag{5a}$$

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We also define

$$\bar{w}_v(n\Delta t) = \bar{w}_{vn} = B_v \exp(-C_v n\Delta t) \sin(E_v n\Delta t + F_v). \tag{5b}$$

Using the trigonometric addition theorem for cosine and sine and the substitutions:

$$R_{v1} = \exp(-C_v\Delta t), \tag{6a}$$

$$S_{v1} = \cos(E_v\Delta t), \tag{6b}$$

$$T_{v1} = \sin(E_v\Delta t), \tag{6c}$$

we can write for the next time interval $t = (n + 1)\Delta t$:

$$w_{v,n+1} = R_{v1}B_v \exp(-C_v n\Delta t)(S_{v1}\cos(E_v n\Delta t + F_v) - T_{v1}\sin(E_v n\Delta t + F_v)) \tag{7}$$

Shifting the time index one unit and using eq. (5), we obtain

$$w_{vn} = R_{v1}(S_{v1}w_{v,n-1} - T_{v1}\bar{w}_{v,n-1}), \tag{8a}$$

and, similarly,

$$\bar{w}_{vn} = R_{v1}(S_{v1}\bar{w}_{v,n-1} + T_{v1}w_{v,n-1}). \tag{8b}$$

Let us now approximate the convolution by both linear and quadratic interpolations.

a. *Linear approximation of the convolution integral.* The linear approximation (trapezoidal rule) for P_{vn} in eq. (4c) is

$$P_{vn} = \Delta t \sum_{m=0}^{n-1} b_m x_m w_{v,n-m} \tag{9a}$$

with

$$b_m = \begin{cases} 0.5 & \text{for } m = 0, \\ 1 & \text{for } m = 1, 2, \cdots, n-1. \end{cases}$$

Similarly, we can define $\bar{P}_{vn}$ with the same coefficients b_m:

$$\bar{P}_{vn} = \Delta t \sum_{m=0}^{n-1} b_m x_m \bar{w}_{v,n-m}. \tag{9b}$$

Now, the objective is to express P_{vn} by the previous value ($P_{v,n-1}$) and a correction. For this purpose we write

$$P_{v,n-1} = \Delta t \sum_{m=0}^{n-2} b_m x_m w_{v,n-1-m}, \tag{10a}$$

and, similarly,

$$\bar{P}_{v,n-1} = \Delta t \sum_{m=0}^{n-2} b_m x_m \bar{w}_{v,n-1-m} \tag{10b}$$

with

$$b_m = \begin{cases} 0.5 & \text{for } m = 0, \\ 1 & \text{for } m = 1, 2, \cdots, n-2. \end{cases}$$

If eq. (8a) is introduced into eq. (9a), using eq. (10) we obtain

$$P_{vn} = R_{v1}(S_{v1}P_{v,n-1} - T_{v1}\bar{P}_{v,n-1}) + \Delta t w_{v1} x_{n-1}, \quad (11a)$$

and, similarly,

$$\bar{P}_{vn} = R_{v1}(S_{v1}\bar{P}_{v,n-1} + T_{v1}P_{v,n-1}) + \Delta t \bar{w}_{v1} x_{n-1}. \quad (11b)$$

To minimize the number of multiplications during the iterations, the following time-independent parameters can be calculated at the beginning of the iterations:

$$U_{v1} = R_{v1}S_{v1} \quad (12a)$$

$$V_{v1} = R_{v1}T_{v1} \quad (12b)$$

$$W_{v1} = \Delta t w_{v1} \quad (12c)$$

$$\bar{W}_{v1} = \Delta t \bar{w}_{v1} \quad (12d)$$

$$Q_{v1} = \Delta t b_n w_{v0} \quad \text{with} \quad b_n = 0.5 \quad (12e)$$

where

$$w_{v1} = B_v \exp(-C_v \Delta t) \cos(E_v \Delta t + F_v)$$

$$\bar{w}_{v1} = B_v \exp(-C_v \Delta t) \sin(E_v \Delta t + F_v)$$

$$w_{v0} = B_v \cos F_v$$

Equation (3b) describes an impulse response $w_v(t)$ which corresponds to a Laplace transfer function $W_v(s)$ containing a single complex pole. The pole becomes a real number when the oscillatory portion of eq. (3b) disappears, i.e. when $E_v = F_v = 0$. In this case, we obtain from eq. (6) that $S_{v1} = 1$ and $T_{v1} = 0$ and from eqs. (12a) and (12b) that $U_{v1} = R_{v1}$ and $V_{v1} = 0$.

The final formulas for P_{vn} and $\bar{P}_{vn}$ are

$$P_{vn} = U_{v1}P_{v,n-1} - V_{v1}\bar{P}_{v,n-1} + W_{v1}x_{n-1} \quad (13a)$$

and, similarly,

$$\bar{P}_{vn} = U_{v1}\bar{P}_{v,n-1} + V_{v1}P_{v,n-1} + W_{v1}x_{n-1}. \quad (13b)$$

Hence, if w_v is of the type

$$w_v = B_v \exp(-C_v \Delta t) \cos(E_v t + F_v),$$

the output signal y_n can be calculated using the following simple recursion formula together with eqs. (13a) and (12e):

$$y_n = \sum_{v=1}^{A} P_{vn} + x_n \sum_{v=1}^{A} Q_{v1}. \quad (14)$$

If we approximate the convolution integral using Simpson's rule, we obtain a more accurate formula for P_n, although more calculations have to be performed at the beginning of the iterations.

b. *Quadratic approximation of the convolution integral.* The quadratic approximation (Simpson's rule) distinguishes between the odd and the even samples. If n, the number of samples, is odd, the interpolation between the first two samples

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(for $m = 0$ and $m = 1$) is linear and between all other samples, quadratic. If n is even, the interpolation is always quadratic.

The quadratic approximation of $P_{v,n}$ and $\bar{P}_{v,n}$ can be written as

$$P_{vn} = \frac{2}{3}\Delta t \sum_{m=0}^{n-1} b_m x_m w_{v,n-m} \tag{15a}$$

and, similarly,

$$\bar{P}_{vn} = \frac{2}{3}\Delta t \sum_{m=0}^{n-1} b_m x_m \bar{w}_{v,n-m} \tag{15b}$$

where for even n,

$$b_m = \begin{cases} 0.5, & m = 0, \\ 1, & m = 2, 4, 6, \cdots, n-2, \\ 2, & m = 1, 3, 5, \cdots, n-1, \end{cases}$$

and for odd n,

$$b_m = \begin{cases} 0.75, & m = 0, \\ 1.25, & m = 1, \\ 1, & m = 3, 5, 7, \cdots, n-2, \\ 2, & m = 2, 4, 6, \cdots, n-1. \end{cases}$$

P_{vn} will be expressed by the prior value $P_{v,n-2}$, and not $P_{v,n-1}$. We can then write

$$P_{v,n-2} = \frac{2}{3}\Delta t \sum_{m=0}^{n-3} b_m x_m w_{v,n-2-m}, \tag{16a}$$

and, similarly,

$$\bar{P}_{v,n-2} = \frac{2}{3}\Delta t \sum_{m=0}^{n-3} b_m x_m w_{v,n-2-m}. \tag{16b}$$

As in eq. (8), w_{vn} can be expressed by the prior values $w_{v,n-2}$ and $\bar{w}_{v,n-2}$. Hence

$$w_{vn} = R_{v2}(S_{v2} w_{v,n-2} - T_{v2} \bar{w}_{v,n-2}) \tag{17a}$$

and

$$\bar{w}_{vn} = R_{v2}(S_{v2} \bar{w}_{v,n-2} + T_{v2} w_{v,n-2}) \tag{17b}$$

with

$$R_{v2} = \exp(-2C_v \Delta t), \tag{18a}$$

$$S_{v2} = \cos(2E_v \Delta t), \tag{18b}$$

$$T_{v2} = \sin(2E_v \Delta t). \tag{18c}$$

Introducing these values into eq. (15) and considering eq. (16), we obtain for both even and odd n,

$$P_{vn} = R_{v2}(S_{v2} P_{v,n-2} - T_{v2} \bar{P}_{v,n-2}) + \tfrac{2}{3}\Delta t(x_{n-2} w_{v2} + 2x_{n-1} w_{v1}). \tag{19a}$$

and, similarly,

$$\bar{P}_{vn} = R_{v2}(S_{v2}\bar{P}_{v,n-2} + T_{v2}P_{v,n-2}) + \tfrac{2}{3}\Delta t(x_{n-2}\bar{w}_{v2} + 2x_{n-1}\bar{w}_{v1}). \tag{19b}$$

The time-independent coefficients can be computed once at the beginning of the iterations. They are:

$$U_{v2} = R_{v2}S_{v2} \tag{20a}$$

$$V_{v2} = R_{v2}T_{v2} \tag{20b}$$

$$W_{v2} = \tfrac{2}{3}\Delta t w_{v2} \tag{20c}$$

$$\bar{W}_{v2} = \tfrac{2}{3}\Delta t \bar{w}_{v2} \tag{20d}$$

$$Y_{v2} = \tfrac{4}{3}\Delta t w_{v1} \tag{20e}$$

$$\bar{Y}_{v2} = \tfrac{4}{3}\Delta t \bar{w}_{v1} \tag{20f}$$

$$Q_{v2} = b_n \Delta t w_{v0} \qquad b_n = \begin{cases} 1/2 & \text{for odd } n \\ 1/3 & \text{for even } n \end{cases} \tag{20g}$$

$$w_{v2} = B_v \exp(-2C_v\Delta t) \cos(2E_v\Delta t + F_v)$$

$$\bar{w}_{v2} = B_v \exp(-2C_v\Delta t) \sin(2E_v\Delta t + F_v)$$

The final formulas for P_{vn} and $\bar{P}_{vn}$ are

$$P_{vn} = U_{v2}P_{v,n-2} - V_{v2}\bar{P}_{v,n-2} + W_{v2}x_{n-2} + Y_{v2}x_{n-1} \tag{21a}$$

and, similarly,

$$\bar{P}_{vn} = U_{v2}\bar{P}_{v,n-2} + V_{v2}P_{v,n-2} + \bar{W}_{v2}x_{n-2} + \bar{Y}_{v2}x_{n-1}. \tag{21b}$$

The output signal y_n can be computed by using eqs. (21a) and (20g);

$$y_n = \sum_{v=1}^{A} P_{vn} + x_n \sum_{v=1}^{A} Q_{v2} \tag{22}$$

2.2 Recursion Formulas for Case 2

At $t = n \cdot \Delta t$ the weighting function $w_v^{(D_v)}$ has the value

$$w_{vn}^{(D_v)} = B_v n^{D_v} \Delta t^{D_v}. \tag{23}$$

The next sample is

$$w_{v,n+1}^{(D_v)} = B_v(n+1)^{D_v}\Delta t^{D_v}. \tag{24}$$

Equation (24) can be written in the following from by applying the binomial series

$$w_{v,n+1}^{(D_v)} = B_v \Delta t^{D_v} \sum_{p=0}^{D_v} \binom{D_v}{p} n^p \tag{25}$$

or

$$w_{v,n+1}^{(D_v)} = \sum_{p=0}^{D_v} \binom{D_v}{p} \Delta t^{D_v - p} w_{vn}^{(p)}$$

since

$$w_{vn}^{(p)} = B_v n^p \Delta t^p. \tag{26}$$

Shifting the time index, we simply get

$$w_{vn}^{(D_v)} = \sum_{p=0}^{D_v} \binom{D_v}{p} \Delta t^{D_v - p} w_{v,n-1}^{(p)}. \tag{27}$$

Let us now distinguish between the linear and the quadratic approximation of the convolution.

a. *Linear approximation of the convolution integral.* The linear approximation for $P_{vn}^{(D_v)}$ is

$$P_{vn}^{(D_v)} = \Delta t \sum_{m=0}^{n-1} b_m x_m w_{v,n-m}^{(D_v)} \tag{28a}$$

or

$$P_{vn}^{(D_v)} = \Delta t b_{n-1} x_{n-1} w_{v1}^{(D_v)} + \Delta t \sum_{m=0}^{n-2} b_m x_m w_{v,n-m}^{(D_v)}. \tag{28b}$$

If we introduce eq. (27) into eq. (28b), we obtain

$$P_{vn}^{(D_v)} = \Delta t b_{n-1} x_{n-1} w_{v1}^{(D_v)} + \Delta t \sum_{m=0}^{n-2} b_m x_m \sum_{p=0}^{D_v} \binom{D_v}{p} \Delta t^{D_v - p} w_{v,n-1-m}^{(p)}. \tag{28c}$$

We now interchange the summation symbols and obtain:

$$P_{vn}^{(D_v)} = \Delta t b_{n-1} x_{n-1} w_{v1}^{(D_v)} + \sum_{p=0}^{D_v} \binom{D_v}{p} \Delta t^{D_v - p} \Delta t \sum_{m=0}^{n-2} b_m x_m w_{v,n-1-m}^{(p)}. \tag{29}$$

However, from eq. (28a) we have

$$P_{v,n-1}^{(p)} = \Delta t \sum_{m=0}^{n-2} b_m x_m w_{v,n-1-m}^{(p)}. \tag{28d}$$

Knowing that $b_{n-1} = 1$ and that $w_{v1}^{(D_v)} = B_v \Delta t^{D_v}$ (from eq. (23)), we can write for $D_v > 0$

$$P_{vn}^{(D_v)} = B_v \Delta t^{D_v + 1} x_{n-1} + \sum_{p=0}^{D_v} \binom{D_v}{p} \Delta t^{D_v - p} P_{v,n-1}^{(p)}. \tag{30}$$

In order to evaluate eq. (30),

$$P_{vn}^{(0)} = P_{v,n-1}^{(0)} + B_v \Delta t x_{n-1}$$

must be calculated and then in consecutive order we must calculate

$$P_{v,n-1}^{(1)} = P_{v,n-1}^{(2)} \cdots P_{v,n-1}^{(D_v - 1)}$$

using eq. (30). The output signal y_n becomes

$$y_n = \sum_{v=1}^{A} (P_{vn}^{(D_v)}) + x_n \sum_{v=1}^{A} Q_{v1}. \tag{31}$$

The computational burden becomes rather extensive if D_v is large. In this case it may be more advantageous to perform lower order convolutions sequentially.

b. *Quadratic approximation of the convolution.* The formulas for the quadratic

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approximation of the convolution can be easily derived by applying the knowledge of Sections 2.1b and 2.2a. Since

$$w_{v,n+2}^{(D_v)} = B_v(n+2)^{D_v}\Delta t^{D_v}, \tag{32}$$

by using the binomial series and shifting the time index, we can write

$$w_{vn}^{(D_v)} = \sum_{p=0}^{D_v} 2^{D_v-p}\binom{D_v}{p}\Delta t^{D_v-p} w_{v,n-2}^{(p)}. \tag{33}$$

If we introduce this equation into eq. (15a) and use eq. (16a), we obtain

$$P_{vn}^{(D_v)} = B_v \tfrac{4}{3}\,\Delta t^{D_v+1}(2^{D_v-1}x_{n-2} + x_{n-1}) + \sum_{p=0}^{D_v} 2^{D_v-p}\binom{D_v}{p}\Delta t^{D_v-p} P_{v,n-2}^{(p)}. \tag{34}$$

Hence the output signal y_n is

$$y_n = \sum_{v=1}^{A} (P_{vn}^{(D_v)}) + x_n \sum_{v=1}^{A} Q_{v2}. \tag{35}$$

The equations derived so far can be extended for the general case of $w(t)$, i.e. eq. (3).

2.3. Recursion Formulas for the General Case of $w(t)$

The formulas $P_{vn}^{(D_v)}$ for the general case of $w(t)$ as described by eq. (3) can be arrived at by combining the formulas of Sections 2.1 and 2.2.

a. *Linear approximation of the convolution.* Applying the trigonometric addition theorem for cosine and sine, and the binomial series for $(n+1)^{D_v}$, we can write

$$w_{vn}^{(D_v)} = R_{v1}\sum_{p=0}^{D_v}\binom{D_v}{p}\Delta t^{D_v-p}(S_{v1}w_{v,n-1}^{(p)} - T_{v1}\bar{w}_{v,n-1}^{(p)}) \tag{36a}$$

and, similarly,

$$\bar{w}_{vn}^{(D_v)} = R_{v1}\sum_{p=0}^{D_v}\binom{D_v}{p}\Delta t^{D_v-p}(S_{v1}\bar{w}_{v,n-1}^{(p)} + T_{v1}w_{v,n-1}^{(p)}) \tag{36b}$$

with

$$w_{vn}^{(p)} = B_v(n\Delta t)^p \exp(-C_v n\Delta t)\cos(E_v n\Delta t + F_v) \tag{37a}$$

and, similarly,

$$\bar{w}_{vn}^{(p)} = B_v(n\Delta t)^p \exp(-C_v n\Delta t)\sin(E_v n\Delta t + F_v). \tag{37b}$$

If we introduce eq. (36a) into eq. (9a) and eq. (36b) into eq. (9b), we can write

$$P_{vn}^{(D_v)} = B_v\cos(F_v)\Delta t^{D_v+1}x_{n-1} + R_{v1}\sum_{p=0}^{D_v}\binom{D_v}{p}\Delta t^{D_v-p}(S_{v1}P_{v,n-1}^{(p)} - T_{v1}\bar{P}_{v,n-1}^{(p)}) \tag{38a}$$

and, similarly,

$$\bar{P}_{vn}^{(D_v)} = B_v\sin(F_v)\Delta t^{D_v+1}x_{n-1} + R_{v1}\sum_{p=0}^{D_v}\binom{D_v}{p}\Delta t^{D_v-p}(S_{v1}\bar{P}_{v,n-1}^{(p)} + T_{v1}P_{v,n-1}^{(p)}). \tag{38b}$$

If the time-independent coefficients

$$U_{v1,p} = \binom{D_v}{p}\Delta t^{D_v-p}R_{v1}S_{v1} \tag{39a}$$

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$$V_{si,p} = \binom{D_s}{p} \Delta t^{D_s-p} R_{si} T_{si} \tag{39b}$$

$$W_{si} = w_{si}^{(1)} \Delta t^{D_s+1} \tag{39c}$$

$$\bar{W}_{si} = \bar{w}_{si}^{(1)} \Delta t^{D_s+1} \tag{39d}$$

$$Q_{s1} = \Delta t\, 0.5\, w_{s0} \qquad (Q_{s1} = 0 \text{ for } D_s > 0) \tag{39e}$$

are introduced at the beginning of the iterations, then for $D_s \geq 0$,

$$P_{sn}^{(D_s)} = W_{s1}x_{n-1} + \sum_{p=0}^{D_s} (U_{s1,p}P_{s,n-1}^{(p)} - V_{s1,p}\bar{P}_{s,n-1}^{(p)}) \tag{40a}$$

and, similarly,

$$\bar{P}_{sn}^{(D_s)} = \bar{W}_{s1}x_{n-1} + \sum_{p=0}^{D_s} (U_{s1,p}\bar{P}_{s,n-1}^{(p)} + V_{s1,p}P_{s,n-1}^{(p)}) \tag{40b}$$

The output signal y_n can be computed by using eqs. (40a) and (39e):

$$y_n = \sum_{s=1}^{k} P_{sn}^{(D_s)} + x_n \sum_{s=1}^{k} Q_{s1} \tag{41}$$

b. *Quadratic approximation of the convolution.* By the same procedure that was used in Sections 2.1b and 2.2b, we obtain the following formulas:

$$P_{sn}^{(D_s)} = W_{s2}x_{n-2} + Y_{s2}x_{n-1} + \sum_{p=0}^{D_s} (U_{s2,p}P_{s,n-2}^{(p)} - V_{s2,p}\bar{P}_{s,n-2}^{(p)}) \tag{42a}$$

and, similarly,

$$\bar{P}_{sn}^{(D_s)} = \bar{W}_{s2}x_{n-2} + \bar{Y}_{s2}x_{n-1} + \sum_{p=0}^{D_s} (U_{s2,p}\bar{P}_{s,n-2}^{(p)} + V_{s2,p}P_{s,n-2}^{(p)}) \tag{42b}$$

with

$$U_{s2,p} = 2^{D_s-p}\binom{D_s}{p} \Delta t^{D_s-p} R_{s2} S_{s2} \tag{43a}$$

$$V_{s2,p} = 2^{D_s-p}\binom{D_s}{p} \Delta t^{D_s-p} R_{s2} T_{s2} \tag{43b}$$

$$W_{s2} = \tfrac{1}{3}(2\Delta t)^{D_s+1} w_{s2}^{(2)} \tag{43c}$$

$$Y_{s2} = \tfrac{4}{3}\Delta t^{D_s+1} w_{s2}^{(3)} \tag{43d}$$

$$\bar{W}_{s2} = \tfrac{1}{3}(2\Delta t)^{D_s+1} \bar{w}_{s2}^{(2)} \tag{43e}$$

$$\bar{Y}_{s2} = \tfrac{4}{3}\Delta t^{D_s+1} \bar{w}_{s2}^{(3)} \tag{43f}$$

$$Q_{s2} = \begin{cases} \frac{1}{3}\Delta t w_{s0} & \text{for odd } n \\ \frac{1}{3}\Delta t w_{s0} & \text{for even } n \end{cases} \tag{43g}$$

$$(Q_{s2} = 0 \quad \text{for} \quad D_s > 0)$$

The output signal y_n can then be computed by using eqs. (42a) and (43g) as follows:

$$y_n = \sum_{\nu=1}^{A} P_{\nu n}^{(D_\nu)} + x_n \sum_{\nu=1}^{A} Q_{\nu 2} \tag{44}$$

3. *Summary*

The computation of the output signal y_n by formulas (41) and (44) requires considerably fewer operations and much less storage space than is required to compute y_n by the general formula (eq. (2)).

If we consider a transfer function having only single poles, i.e. $D_\nu = 0$ in eq. (3b) (which is the most frequently occurring case), we compute y_n by the recursion formula (14) or (22). In this case the number of arithmetic operations per time step are counted as follows.

If the linear approximation is used, six multiplications and four additions in eq. (13) and one addition in eq. (14) must be performed per single complex pole, and one multiplication and one addition regardless of the number of poles. If the quadratic approximation is used, eight multiplications and six additions in eq. (21) and one addition in eq. (22) must be performed per single complex pole, and one multiplication and one addition regardless of the number of poles. The number of multiplications and additions reduces for real poles since in this case $V_{\nu 1}$ and $V_{\nu 2}$ are both zero.

Let us compare the number of operations and storage space necessary to compute y_n by the two methods for one thousand samples of a transfer function having two damped oscillations and one exponential function.

When the linear approximation is applied, the recursion formula eq. (14) together with eq. (13) requires $(2 \times 6 + 4 + 1)1000 = 17{,}000$ multiplications and $(2 \times 5 + 3 + 1)1000 = 14{,}000$ additions.

Only seven past values, i.e. x_{n-1}, $P_{1,n-1}$, $\bar{P}_{1,n-1}$, $\cdots$, $\bar{P}_{3,n-1}$ plus the constant parameters U_{11}, V_{11}, W_{11}, $\bar{W}_{11}$, Q_{11}; U_{31}, $\cdots$, Q_{31} need to be stored. On the other hand, the general formula eq. (2) requires approximately $n \times n/2$, i.e. 500,000 multiplications and additions, for $n = 1000$, and storage space for 1000 values of x and w must be reserved.

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